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Thermal Buckling of Functionally Graded Plates

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Equilibrium and stability equations of a rectangular plate made of functionally graded material under thermal loads are derived, based on the classical plate theory. When it is assumed that the material properties vary as a power form of thickness coordinate variable z and when the variational method is used, the system of fundamental differential equations is established. The derived equilibrium and stability equations for functionally graded plates are identical with the equations for homogeneous plates. Buckling analysis of functionally graded plates under four types of thermal loads is carried out resulting in closed-form solutions. The buckling loads are reduced to the critical buckling temperature relations for functionally graded plates with linear composition of constituent materials and homogeneous plates. The results are validated with the reduction of the buckling relations for functionally graded plates to those of isotropic homogeneous plates given in the literature.

Nomenclature

a, b	=	plate length and width, respectively
$E(z), E_c, E_m$	=	elasticity modulus of the functionally graded material (FGM), ceramic, and metal
h	=	plate thickness
k	=	power law index
k_x, k_y, k_{xy}	=	curvature changes and twist
M_x, M_y, M_{xy}	=	bending and twisting moment intensities
m, n	=	number of half waves in x and y directions, respectively
N_x, N_y, N_{xy}	=	normal and shearing force intensities
N_{x0}, N_{y0}, N_{xy0}	=	prebuckling force resultants
$T(x, y, z)$	=	temperature distribution
T_c, T_m	=	temperature at the ceramic-rich and metal-rich surfaces of the plate
T_0, T_a	=	temperature at $x = 0$ and a, respectively
U	=	strain energy
U_m, U_b, U_c, U_T	=	membrane, bending, coupled, and thermal strain energies
u, v, w	=	displacement components in the rectangular coordinates
V	=	total potential energy
V_c, V_m	=	volume fractions of the ceramic and metal
x, y, z	=	rectangular Cartesian coordinates
$\alpha(z), \alpha_c, \alpha_m$	=	coefficient of thermal expansion of the FGM, ceramic, and metal
β_x, β_y	=	rotations of plate element relative to the y and x coordinate directions, respectively
γ_{xy}	=	shear strain at the middle surface of the plate
$\bar{\gamma}_{xy}$	=	shear strain at any point through the plate thickness
$\Delta T_{\rm cr}$	=	critical buckling temperature difference
ϵ_x, ϵ_y	=	normal strains at the middle surface of the plate
$\bar{\epsilon}_x, \bar{\epsilon}_y$	=	normal strains at any point through the plate thickness
ν_0	=	Poisson's ratio
$\bar{\sigma}_x, \bar{\sigma}_y$	=	normal stresses at any point through the plate thickness
$\bar{\tau}_{xy}$	=	shear stress at any point through the plate thickness

Introduction

THE nonlinear equilibrium equations and associated linear stability equations were expressed for bars, plates, and shells by Brush and Almroth in 1975.[1] The subject matter of Ref. 1 is the buckling behavior of structural members subjected to mechanical loads. Subsequently, many researchers developed equilibrium and stability equations for plates and shells made of composite layered materials and used them for determination of buckling and vibrational behavior of structures.

A review of recent developments in laminated composite plate buckling was carried out by Leissa[2] and Tauchert.[3] Recently many research works have been directed toward the buckling analysis of composite plates under mechanical and thermal loads.[4–13] More advance solutions including transverse shear effects have been proposed by other researchers.[14–23]

Functionally graded materials (FGMs) are advanced high-performance, heat-resistant materials able to withstand ultrahigh temperatures and extremely large gradients present in spacecraft and nuclear plants. FGMs are microscopically inhomogeneous in which the mechanical properties vary smoothly and continuously from one surface to the other. This is achieved by gradually varying the volume fraction of the constituent materials.[24] These novel materials were first introduced by a group of scientists in Sendai, Japan, in 1984[25] and then were developed rapidly by other scientists.[26,27]

It is apparent from the literature survey that most of the research on FGMs have been restricted to thermal stress analysis, fracture mechanics, and optimization. Very little work has been done to consider the stability analysis, buckling, and vibrational behavior of FGM structures. Some research works related to the present study are introduced in the following.

Tanigawa et al.[28] derived a one-dimensional temperature solution for a nonhomogeneous plate in transient state and also optimized the material composition by introducing a laminated composite model. Analytical formulation and numerical solution of the thermal stresses and deformations for axisymmetrical shells of FGM subjected to thermal loading due to fluid is obtained by Takezono et al.[29] The temperature distribution through the thickness was assumed to be a curve of high order, and the temperature field in the shell was determined using the equation of heat conduction. The equations of equilibrium and the relations between the strains and displacements were derived from Sander's elastic shell theory.

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A new higher-order theory for functionally graded materials that explicitly couples the microstructural and macrostructural effects was developed by Aboudi et al.[30] As a practical consequence, functionally graded plates have been analyzed in the presence of imposed average normal stresses in the nongraded in-plane directions. The constitutive equation of particle-reinforced composites with consideration of the damage process and change in temperature have been estimated by the equivalent inclusion method by Asakawa et al.[31] The response of a functionally graded ceramic–metal plate is investigated by Praveen and Reddy[32] using a plate finite element model that accounts for the transverse shear strains, rotary inertia, and moderately large rotations in the von Kármán sense. The static and dynamic thermoelastic responses of the functionally graded plates are investigated using a simple power law distribution for the volume fraction. The governing differential equations of motion were derived from Hamilton's principle. A recently developed micromechanical theory for the thermoelastic response of functionally graded composites with nonuniform fiber spacing through thickness is extended by Aboudi et al.[33] The optimal composition profile problems of the FGMs to decrease the thermal stresses and thermal stress intensity factor are discussed by Noda.[34] He concluded that, when the continuously changing composition between ceramics and metals can be selected pertinently, thermal stresses in the FGMs is drastically decreased.

Sumi[35] studied the propagation and reflection of thermal and mechanical waves in FGMs under impulsive heat addition. Development of the analysis is based on the equations of coupled thermoelasticity along with a modified Fourier's law. The dynamic thermoelastic response of functionally graded cylinders and plates was studied by Reddy and Chin.[36] They derived a thermoelastic boundary value problem using the first-order shear deformation plate theory (FSDT) that accounts for transverse shear strains and rotations coupled with a three-dimensional heat conduction equation for a functionally graded plate. Zimmerman and Lutz[37] determined the exact solution of thermal stresses and thermal expansions for a uniformly heated functionally graded cylinder. They derived a governing equation for the thermoelastic equilibrium of the cylinder, by substituting stress–strain and kinematic relations into the stress equilibrium equation of cylinder allowing the elastic moduli and thermal expansion coefficient to vary with the radial coordinate. Then, the equation was solved analytically.

Tanigawa et al.[38] have treated the three-dimensional thermoelastic problem of a medium with nonhomogeneous material properties such as shear modulus of elasticity, coefficient of linear thermal expansion, and thermal conductivity. They proposed a power law distribution along the thickness for these material properties and established the fundamental equations by means of three kinds of displacement functions. For illustration, they considered the thermoelastic problem of a semi-infinite body. Birman[39] studied the buckling problem of functionally graded composite rectangular plates subjected to uniaxial compression. Two classes of fibers were used in the hybrid composite material. Linear equations of equilibrium of a symmetrically laminated plate, which are uncoupled, were derived and solved to obtain the critical buckling load for the simply supported condition.

In the present paper, equilibrium and stability equations for the rectangular functionally graded plates are obtained on the basis of classical plate theory. Resulting equations are employed to obtain the closed-form solutions for the critical buckling temperature. To establish the fundamental system of equations for the buckling analysis, it is assumed that the nonhomogeneous mechanical and thermal properties are given by a power form of coordinate variable z.

Functionally Graded Plates

FGMs are typically made from a mixture of ceramics and metal or a combination of different metals. The ceramic constituent of the material provides the high-temperature resistance due to its low thermal conductivity. The ductile metal constituent prevents fracture caused by stresses due to the high-temperature gradient in a very short period of time. Furthermore, a mixture of a ceramic and a metal with a continuously varying volume fraction can be easily manufactured.

The volume fractions of the ceramic V_c and metal V_m corresponding to the power law are expressed as[32,36]

$$V_c = [(2z + h)/2h]^k, \qquad V_m = 1 - V_c \tag{1}$$

where z is the thickness coordinate; and $-h/2 \le z \le h/2$, where h is the thickness of the plate and k is the power law index that takes values greater than or equal to zero. The variation of the composition of ceramics and metal is linear for $k = 1$. The value of k equal to zero represents a fully ceramic plate. The mechanical and thermal properties of FGMs are determined from the volume fraction of the material constituents. We assume that the nonhomogeneous material properties such as the modulus of elasticity E and the coefficient of thermal expansion α change in the thickness direction z based on the Voigt's rule over the whole range of the volume fraction,[35,40] whereas Poisson's ratio ν is assumed to be constant[38,41] as

$$E(z) = E_c V_c + E_m(1 - V_c)$$
$$\alpha(z) = \alpha_c V_c + \alpha_m(1 - V_c), \qquad \nu(z) = \nu_0 \tag{2}$$

where subscripts m and c refer to the metal and ceramic constituents, respectively. When Eqs. (1) are substituted into Eqs. (2), material properties of the FGM plate are determined, which are the same as the equations proposed by Praveen and Reddy[32]:

$$E(z) = E_m + E_{cm}[(2z + h)/2h]^k$$
$$\alpha(z) = \alpha_m + \alpha_{cm}[(2z + h)/2h]^k, \qquad \nu(z) = \nu_0 \tag{3}$$

where

$$E_{cm} = E_c - E_m, \qquad \alpha_{cm} = \alpha_c - \alpha_m \tag{4}$$

Equilibrium and Stability Equations

We initially consider a FGM rectangular thin flat plate of length a, width b, and thickness h, subjected to the thermal loads. Rectangular Cartesian coordinates x, y, and z are assumed for derivations.

The analyses of plates are often based on classical plate theory (CPT). This theory is based on the Love–Kirchhoff hypothesis and assumes that 1) the straight lines do not undergo axial deformation, 2) the straight lines perpendicular to the midsurface before deformation remain straight after deformation, and 3) the straight lines rotate such that they remain perpendicular to the midsurface after deformation. The first two assumptions imply that the transverse displacement is independent of the transverse coordinate and the transverse normal strain is zero. The third assumption results in zero transverse shear strains. Thus, in the CPT all transverse stresses are neglected. In the present study, CPT is used to obtain the equilibrium and stability equations as well as critical buckling relations.

The strains across the plate thickness at a distance z from the middle surface are[1,13]

$$\bar{\epsilon}_x = \epsilon_x + zk_x, \qquad \bar{\epsilon}_y = \epsilon_y + zk_y, \qquad \bar{\gamma}_{xy} = \gamma_{xy} + 2zk_{xy} \tag{5}$$

where ϵ_x and ϵ_y are the normal strains, γ_{xy} is the shear strain at the middle surface of the plate, and k_{ij} are the curvatures.

According to Sander's assumption (see Ref. 1), the general nonlinear strain–displacement relations can be simplified to give the following terms for the strains at the middle surface and the curvatures in terms of the displacement components u, v, and w in the rectangular coordinates:

$$\epsilon_x = u_{,x} + \tfrac{1}{2}\beta_x^2, \qquad \epsilon_y = v_{,y} + \tfrac{1}{2}\beta_y^2, \qquad \gamma_{xy} = (u_{,y} + v_{,x}) + \beta_x\beta_y$$
$$k_x = \beta_{x,x}, \qquad k_y = \beta_{y,y}, \qquad k_{xy} = \tfrac{1}{2}(\beta_{x,y} + \beta_{y,x})$$
$$\beta_x = -w_{,x}, \qquad \beta_y = -w_{,y} \tag{6}$$

where (,) indicates the partial derivative. Hooke's law for a plate is defined as

$$\bar{\sigma}_x = [E/(1 - \nu^2)][\bar{\epsilon}_x + \nu\bar{\epsilon}_y - (1 + \nu)\alpha T]$$
$$\bar{\sigma}_y = [E/(1 - \nu^2)][\bar{\epsilon}_y + \nu\bar{\epsilon}_x - (1 + \nu)\alpha T]$$
$$\bar{\tau}_{xy} = [E/2(1 + \nu)]\bar{\gamma}_{xy} \tag{7}$$

where E, ν, and α are elastic modulus, Poisson's ratio, and thermal expansion coefficient, respectively.

The forces and moments per unit length of plate expressed in terms of the stress components through the thickness are

$$N_{ij} = \int_{-h/2}^{h/2} \bar{\sigma}_{ij}\,\mathrm{d}z, \qquad M_{ij} = \int_{-h/2}^{h/2} \bar{\sigma}_{ij} z\,\mathrm{d}z \tag{8}$$

Substituting Eqs. (3), (5), and (7) into Eqs. (8) gives the constitutive relations as

$$\begin{aligned}
N_x &= \left[E_1/\left(1-\nu_0^2\right)\right](\epsilon_x + \nu_0\epsilon_y) + \left[E_2/\left(1-\nu_0^2\right)\right](k_x + \nu_0 k_y) \\
&\quad - \Phi_m/(1-\nu_0) \\
N_y &= \left[E_1/\left(1-\nu_0^2\right)\right](\epsilon_y + \nu_0\epsilon_x) + \left[E_2/\left(1-\nu_0^2\right)\right](k_y + \nu_0 k_x) \\
&\quad - \Phi_m/(1-\nu_0) \\
N_{xy} &= [E_1/2(1+\nu_0)]\gamma_{xy} + [E_2/(1+\nu_0)]k_{xy} \\
M_x &= \left[E_2/\left(1-\nu_0^2\right)\right](\epsilon_x + \nu_0\epsilon_y) + \left[E_3/\left(1-\nu_0^2\right)\right](k_x + \nu_0 k_y) \\
&\quad - \Phi_b/(1-\nu_0) \\
M_y &= \left[E_2/\left(1-\nu_0^2\right)\right](\epsilon_y + \nu_0\epsilon_x) + \left[E_3/\left(1-\nu_0^2\right)\right](k_y + \nu_0 k_x) \\
&\quad - \Phi_b/(1-\nu_0) \\
M_{xy} &= [E_2/2(1+\nu_0)]\,\gamma_{xy} + [E_3/(1+\nu_0)]k_{xy}
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
E_1 &= E_m h + \frac{E_{cm} h}{k+1} \\
E_2 &= E_{cm} h^2 \left(\frac{1}{k+2} - \frac{1}{2k+2}\right) \\
E_3 &= \frac{E_m h^3}{12} + E_{cm} h^3 \left[\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right] \\
\Phi_m &= \int_{-h/2}^{h/2} \left[E_m + E_{cm}\left(\frac{2z+h}{2h}\right)^k\right]\left[\alpha_m + \alpha_{cm}\left(\frac{2z+h}{2h}\right)^k\right] \\
&\quad \times T(x,y,z)\,\mathrm{d}z \\
\Phi_b &= \int_{-h/2}^{h/2} \left[E_m + E_{cm}\left(\frac{2z+h}{2h}\right)^k\right]\left[\alpha_m + \alpha_{cm}\left(\frac{2z+h}{2h}\right)^k\right] \\
&\quad \times T(x,y,z) z\,\mathrm{d}z
\end{aligned} \tag{10}$$

The total potential energy of a plate subjected to thermal loads is defined as

$$V = U_m + U_b + U_c + U_T \tag{11}$$

where U_m is membrane strain energy, U_b is bending strain energy, U_c is coupled strain energy, and U_T is thermal strain energy. The strain energy for a thin plate based on the CPT is defined as[42,43]

$$U = \frac{1}{2}\iiint [\bar{\sigma}_x(\bar{\epsilon}_x - \alpha T) + \bar{\sigma}_y(\bar{\epsilon}_y - \alpha T) + \bar{\tau}_{xy}\bar{\gamma}_{xy}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{12}$$

The equilibrium equations of plates may be obtained by the variational approach. When Eqs. (5) and (7) are substituted into Eq. (12), the strain energy components are derived. Then, by setting them into the expression for the total potential energy function (11), with the aid of constitutive law (9), and applying the Euler equations (see Ref. 1), the general equilibrium equations are obtained as

$$\begin{gathered}
N_{x,x} + N_{xy,y} = 0, \qquad N_{xy,x} + N_{y,y} = 0 \\
M_{x,xx} + M_{y,yy} + 2M_{xy,xy} - N_x\beta_{x,x} - N_y\beta_{y,y} - 2N_{xy}\beta_{x,y} = 0
\end{gathered} \tag{13}$$

The stability equations of thin plates may be derived by the variational approach. If V is the total potential energy of the plate, as sum of strain energies, the expansion of V about the equilibrium state by Taylor series is

$$\Delta V = \delta V + \tfrac{1}{2!}\delta^2 V + \tfrac{1}{3!}\delta^3 V + \cdots \tag{14}$$

The first variation δV is associated with the state of equilibrium. The stability of original configuration of the plate in the neighborhood of the equilibrium state can be determined by the sign of second variation $\delta^2 V$. If $\delta^2 V > 0$ for all virtual displacements, the state of equilibrium is stable. The state of equilibrium is unstable if $\delta^2 V < 0$ for at least one admissible set of virtual displacements. However, the condition of $\delta^2 V = 0$ is used to derive the stability equations of many practical plate buckling problems. Thus, the stability equations are represented by the Euler equations for the integrand in the second variation expression[1]:

$$\begin{aligned}
&N_{x1,x} + N_{xy1,y} = 0, \qquad N_{xy1,x} + N_{y1,y} = 0 \\
&M_{x1,xx} + M_{y1,yy} + 2M_{xy1,xy} - N_{x0}\beta_{x1,x} - N_{y0}\beta_{y1,y} \\
&\quad - N_{xy0}(\beta_{x1,y} + \beta_{y1,x}) = 0
\end{aligned} \tag{15}$$

The subscript 1 refers to the state of stability, and the subscript 0 refers to the state of equilibrium conditions. N_{x0}, N_{y0}, and N_{xy0} are the prebuckling force resultants obtained from Eqs. (13). The equilibrium and stability equations of the functionally graded plate are expressed by Eqs. (13) and (15), respectively. These equations are identical with the equations for homogeneous plates, reported previously.[1] However, definition of forces and moments, from Eqs. (9), are different. The material properties are defined by Eqs. (10) for constitution equation of the form of Eqs. (3).

Buckling of Functionally Graded Plates Under Uniform Temperature Rise

The initial uniform temperature of the plate is assumed to be T_i. The plate is simply supported in bending and rigidly fixed in extension. Under these boundary conditions, temperature can be uniformly raised to a final value T_f such that plate buckles.[13] To find the critical $\Delta T = T_f - T_i$, the prebuckling thermal stresses should be found. Solving the membrane form of equilibrium equations, using the method developed by Meyers and Hyer[10] in conjunction with Galerkin's formulation, gives the prebuckling force resultants

$$\begin{gathered}
N_{x0} = -[\Phi_m/(1-\nu_0)], \qquad N_{y0} = -[\Phi_m/(1-\nu_0)] \\
N_{xy0} = 0
\end{gathered} \tag{16}$$

Substituting Eqs. (16) into the stability equations (15) and using the kinematic and constitutive relations results in the buckling equation

$$\frac{E_2^2 - E_1 E_3}{\left(1-\nu_0^2\right)E_1}\nabla^4 w_1 - \frac{\Phi_m}{1-\nu_0}(w_{1,xx} + w_{1,yy}) = 0 \tag{17}$$

The simply supported boundary condition is defined as

$$\begin{aligned}
w_1 = M_{x1} = 0 &\qquad \text{on} \qquad x = 0, a \\
w_1 = M_{y1} = 0 &\qquad \text{on} \qquad y = 0, b
\end{aligned} \tag{18}$$

The following approximate solution is seen to satisfy both the differential equation and the boundary conditions:

$$w_1 = C\sin(m\pi x/a)\sin(n\pi y/b), \qquad m, n = 1, 2, \ldots \tag{19}$$

where m and n are number of half waves in the x and y directions, respectively, and C is a constant coefficient. Substituting Eq. (19) into Eq. (17), and substituting for the thermal parameter Φ_m from Eqs. (10), yields

$$\Delta T = D\left[(mB_a)^2 + n^2\right] \tag{20}$$

$$D = \frac{\pi^2\left(\bar{E}_1\bar{E}_3 - \bar{E}_2^2\right)}{(1+\nu_0)B_h^2\bar{E}_1\{E_m\alpha_m + [1/(k+1)](E_m\alpha_{cm} + E_{cm}\alpha_m) + [1/(2k+1)]E_{cm}\alpha_{cm}\}} \tag{21}$$

where

$$B_a = b/a, \qquad B_h = b/h, \qquad \bar{E}_1 = E_1/h$$
$$\bar{E}_2 = E_2\big/h^2, \qquad \bar{E}_3 = E_3\big/h^3 \tag{22}$$

The critical temperature difference is obtained for the values of m and n that make the preceding expression a minimum. When minimization methods are used, satisfying the Kuhn–Tucker conditions (see Ref. 44), critical temperature difference is obtained for $m = n = 1$. Thus,

$$\Delta T_{\mathrm{cr}} = D\left[(B_a)^2 + 1\right] \tag{23}$$

When the power law index is set equal to one ($k = 1$), Eq. (23) is reduced to the critical temperature difference for functionally graded plate with linear composition of ceramics and metal. Also, when the power law index is set equal to zero ($k = 0$), Eq. (23) is reduced to the critical temperature difference of homogeneous plates:

$$\Delta T_{\mathrm{cr}} = \frac{\pi^2}{12(1+\nu)B_h^2\alpha}\left[(B_a)^2 + 1\right] \tag{24}$$

Equation (24) has been obtained by Tauchert[14] and Thornton[45] for homogeneous isotropic plates.

Buckling of Functionally Graded Plates Subjected to Linear Temperature Change Across the Thickness

Assume a linear temperature variation across the plate thickness as[13]

$$T(z) = (\Delta T/h)(z + h/2) + T_m \tag{25}$$

where z is measured from the middle plane of the plate and T_m is the metal temperature. To find the critical $\Delta T = T(h/2) - T(-h/2)$, the prebuckling thermal stresses should be found. As with the preceding loading cases, solving the membrane form of equilibrium equations results in the prebuckling force resultants expressed by Eqs. (16). Substituting Eqs. (16) into the stability equations (15) and using the kinematic and constitutive relations results in the same buckling equation (17). Substituting approximate solution from Eq. (19) into Eq. (17) and setting the thermal parameter Φ_m for this loading case from Eqs. (10) yields the buckling temperature relation. The critical buckling temperature is obtained when $m = n = 1$. Thus,

$$\Delta T_{\mathrm{cr}} = \frac{1}{L_1}\left\{\frac{\pi^2\left(\bar{E}_1\bar{E}_3 - \bar{E}_2^2\right)}{(1+\nu_0)B_h^2\bar{E}_1}\left[(B_a)^2 + 1\right] - L_2\right\} \tag{26}$$

where

$$L_1 = E_m\alpha_m/2 + [1/(k+2)](E_m\alpha_{cm} + E_{cm}\alpha_m) + [1/(2k+2)]E_{cm}\alpha_{cm}$$
$$L_2 = T_m\{E_m\alpha_m + [1/(k+1)](E_m\alpha_{cm} + E_{cm}\alpha_m) + [1/(2k+1)]E_{cm}\alpha_{cm}\} \tag{27}$$

For $k = 1$, Eq. (26) is reduced to the critical temperature difference for functionally graded plate with linear composition of ceramics and metal. Also, when the power law index is set equal to zero ($k = 0$), Eq. (26) is reduced to the critical temperature difference of homogeneous plates:

$$\Delta T_{\mathrm{cr}} = \frac{\pi^2}{6(1+\nu)B_h^2\alpha}\left[(B_a)^2 + 1\right] - 2T_m \tag{28}$$

Buckling of Functionally Graded Plates Subjected to Nonlinear Temperature Change Across the Thickness

For a functionally graded plate, the coefficient of thermal conduction K is a function of thickness direction z. Similar to Eqs. (3), we assume that the nonhomogeneous property of thermal conductivity may be described in terms of the variable z in a power form as

$$K(z) = K_m + K_{cm}[(2z+h)/2h]^k \tag{29}$$

where

$$K_{cm} = K_c - K_m \tag{30}$$

The steady-state heat conduction equation and the boundary conditions across the plate thickness are

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[K(z)\frac{\mathrm{d}T}{\mathrm{d}z}\right] = 0, \qquad T = T_c, \qquad z = \frac{h}{2}$$
$$T = T_m, \qquad z = -\frac{h}{2} \tag{31}$$

Substituting Eq. (29) into Eqs. (31) results in a differential equation for temperature. The solution is obtained by means of polynomial series. Taking the first seven terms of the series, the solution for temperature distribution across the plate thickness becomes

$$\begin{aligned} T(z) = T_m + \frac{\Delta T}{C}\Bigg[&\left(\frac{2z+h}{2h}\right) - \frac{K_{cm}}{(k+1)K_m}\left(\frac{2z+h}{2h}\right)^{k+1} \\ &+ \frac{K_{cm}^2}{(2k+1)K_m^2}\left(\frac{2z+h}{2h}\right)^{2k+1} - \frac{K_{cm}^3}{(3k+1)K_m^3}\left(\frac{2z+h}{2h}\right)^{3k+1} \\ &+ \frac{K_{cm}^4}{(4k+1)K_m^4}\left(\frac{2z+h}{2h}\right)^{4k+1} - \frac{K_{cm}^5}{(5k+1)K_m^5}\left(\frac{2z+h}{2h}\right)^{5k+1}\Bigg] \end{aligned} \tag{32}$$

with

$$\begin{aligned} C = 1 &- \frac{K_{cm}}{(k+1)K_m} + \frac{K_{cm}^2}{(2k+1)K_m^2} - \frac{K_{cm}^3}{(3k+1)K_m^3} \\ &+ \frac{K_{cm}^4}{(4k+1)K_m^4} - \frac{K_{cm}^5}{(5k+1)K_m^5} \end{aligned} \tag{33}$$

where $\Delta T = T_c - T_m$ is defined as the temperature difference between ceramic-rich and metal-rich surfaces of plate. The prebuckling force resultants for this case of loading is given by Eqs. (16). When the approximate solution (19) is substituted into Eq. (17) and the definition of parameter Φ_m from Eqs. (10) are used, the expression for thermal buckling of the plate is obtained. The critical buckling temperature is computed for $m = n = 1$ as

$$\Delta T_{\mathrm{cr}} = \frac{1}{H_1}\left\{\frac{\pi^2\left(\bar{E}_1\bar{E}_3 - \bar{E}_2^2\right)}{(1+\nu_0)B_h^2\bar{E}_1}\left[(B_a)^2 + 1\right] - H_2\right\} \tag{34}$$

where

$$\begin{aligned} H_1 = \frac{1}{C}\Bigg\{&\frac{E_m\alpha_m}{2} + \frac{1}{k+2}\left[(E_m\alpha_{cm} + E_{cm}\alpha_m) - \frac{E_m\alpha_m K_{cm}}{(k+1)K_m}\right] \\ &+ \frac{1}{2k+2}\Bigg[E_{cm}\alpha_{cm} - \frac{K_{cm}}{(k+1)K_m}(E_m\alpha_{cm} + E_{cm}\alpha_m) \\ &+ \frac{E_m\alpha_m K_{cm}^2}{(2k+1)K_m^2}\Bigg] + \frac{1}{3k+2}\left[\frac{K_{cm}^2}{(2k+1)K_m^2}(E_m\alpha_{cm} + E_{cm}\alpha_m)\right. \end{aligned}$$

$$-\frac{E_m\alpha_m K_{cm}^3}{(3k+1)K_m^3}-\frac{K_{cm}}{(k+1)K_m}E_{cm}\alpha_{cm}\Bigg]+\frac{1}{4k+2}\left[\frac{E_m\alpha_m K_{cm}^4}{(4k+1)K_m^4}\right.$$

$$\left.-\frac{K_{cm}^3}{(3k+1)K_m^3}(E_m\alpha_{cm}+E_{cm}\alpha_m)+\frac{K_{cm}^2}{(2k+1)K_m^2}E_{cm}\alpha_{cm}\right]$$

$$+\frac{1}{5k+2}\left[\frac{K_{cm}^4}{(4k+1)K_m^4}(E_m\alpha_{cm}+E_{cm}\alpha_m)-\frac{E_m\alpha_m K_{cm}^5}{(5k+1)K_m^5}\right.$$

$$\left.-\frac{K_{cm}^3}{(3k+1)K_m^3}E_{cm}\alpha_{cm}\right]+\frac{1}{6k+2}\left[\frac{K_{cm}^4}{(4k+1)K_m^4}E_{cm}\alpha_{cm}\right.$$

$$\left.-\frac{K_{cm}^5}{(5k+1)K_m^5}(E_m\alpha_{cm}+E_{cm}\alpha_m)\right]$$

$$\left.-\frac{1}{7k+2}\left[\frac{K_{cm}^5}{(5k+1)K_m^5}E_{cm}\alpha_{cm}\right]\right\},\qquad H_2=L_2 \tag{35}$$

For $k = 1$, Eq. (34) is reduced to the critical temperature difference for functionally graded plate with linear composition of ceramics and metal. Also, by setting the power law index equal to zero ($k = 0$), Eq. (34) is reduced to Eq. (28) for the critical temperature difference of homogeneous plates.

Buckling of Functionally Graded Plates Subjected to Linear Temperature Change Through the Length

Assume a linear temperature variation along the axial x direction as[13]

$$T(x) = \Delta T(x/a) + T_0 \tag{36}$$

where $\Delta T = T_a - T_0$ is the temperature difference between two ends of a plate at $x = 0$ and a. Substituting Eq. (36) into Eqs. (10) gives the thermal parameter Φ_m as

$$\Phi_m = \phi x + \psi$$

$$\phi = (\Delta T h/a)\{E_m\alpha_m + [1/(k+1)](E_m\alpha_{cm} + E_{cm}\alpha_m)$$
$$+ [1/(2k+1)]E_{cm}\alpha_{cm}\}$$

$$\psi = T_0 h\{E_m\alpha_m + [1/(k+1)](E_m\alpha_{cm} + E_{cm}\alpha_m)$$
$$+ [1/(2k+1)]E_{cm}\alpha_{cm}\} \tag{37}$$

To obtain the critical temperature difference, the prebuckling thermal stresses should be found. Solving the membrane form of the equilibrium equations, using the method developed by Meyers and Hyer[10] in conjunction with Galerkin's method, results in the prebuckling force resultants given as

$$N_{x0} = -\frac{\phi a}{2(1-\nu_0)} - \frac{\psi}{1-\nu_0}$$

$$N_{y0} = -\phi x - \frac{\phi a \nu_0}{2(1-\nu_0)} - \frac{\psi}{1-\nu_0}, \qquad N_{xy0} = 0 \tag{38}$$

Substituting prebuckling forces into the stability equations (15) and using the kinematic and constitutive relations results in the buckling equation

$$\frac{E_2^2 - E_1E_3}{\left(1-\nu_0^2\right)E_1}\nabla^4 w_1 - \frac{\phi a + 2\psi}{2(1-\nu_0)}w_{1,xx}$$

$$-\left[\phi x + \frac{\phi a\nu_0 + 2\psi}{2(1-\nu_0)}\right]w_{1,yy} = 0 \tag{39}$$

Substituting approximate solution from Eq. (19) into the preceding equation, setting the values of ϕ and ψ from Eqs. (37), and applying the Galerkin method yields the buckling temperature relation. Buckling occurs for $m = n = 1$. Calculated ΔT_{cr} is twice the ΔT_{cr} for the buckling of a plate under the uniform temperature rise found from Eq. (23) $-2T_0$, that is,

$$\Delta T_{cra} = 2 \times \Delta T_{cru} - 2T_0 \tag{40}$$

where ΔT_{cra} is the axial buckling temperature and ΔT_{cru} is the buckling temperature under uniform temperature rise. Critical temperature differences for the functionally graded plate with linear composition of ceramics and metal and for homogeneous plates are obtained by substituting from Eqs. (23) and (24) into Eq. (40).

Illustration

To illustrate the proposed approach, a ceramic–metal functionally graded plate is considered. The combination of materials consist of aluminum and alumina. Young's modulus and the coefficient of thermal expansion for aluminum are $E_m = 70$ Gpa and $\alpha_m = 23 \times 10^{-6}$/°C and for alumina are $E_c = 380$ Gpa and $\alpha_c = 7.4 \times 10^{-6}$/°C, respectively. Poisson's ratio is chosen to be 0.3. The plate is assumed to be simply supported on all four edges. Variation of the critical temperature difference ΔT_{cr} vs the dimensionless geometrical parameters b/a and b/h are plotted for three loading cases in Figs. 1–6. In Figs. 1–6, three arbitrary values of

Table 1 Critical buckling temperature difference due to the linear and nonlinear temperature distribution with respect to *k* and *b/a*

k	$b/a = 1$	$b/a = 2$	$b/a = 3$	$b/a = 4$	$b/a = 5$
0					
Linear	24.1982	75.4955	160.9911	280.6848	434.5768
Nonlinear	24.1982	75.4955	160.9911	280.6848	434.5768
1					
Linear	5.5209	27.8683	65.1140	117.2580	184.3003
Nonlinear	7.6635	38.6838	90.3842	162.7649	255.8257
5					
Linear	3.8999	22.6595	53.9256	97.6981	153.9770
Nonlinear	4.8774	28.3389	67.4414	122.1849	192.5694

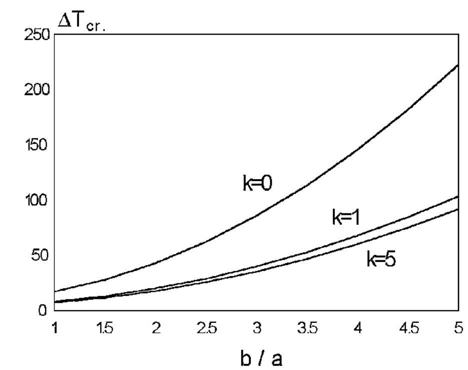

Fig. 1 Critical buckling temperature of the functionally graded plate under uniform temperature rise vs *b/h* (*b/h* = 100).

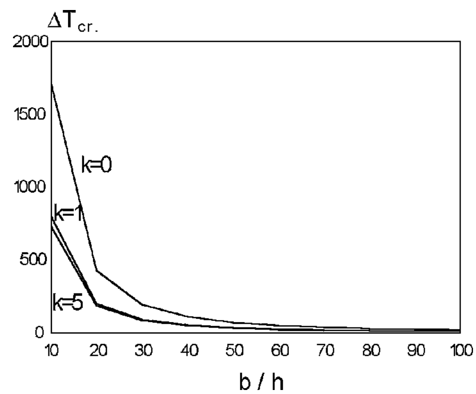

Fig. 2 Critical buckling temperature of the functionally graded plate under uniform temperature rise vs *b/h* (*b/a* = 1).

Table 2 Critical buckling temperature difference due to the linear and nonlinear temperature distribution respect to k and b/h

k		$b/h = 10$	$b/h = 20$	$b/h = 40$	$b/h = 60$	$b/h = 80$	$b/h = 100$
0							
	Linear	3409.8210	844.9553	203.7388	84.9950	43.4347	24.1982
	Nonlinear	3409.8210	844.9553	203.7388	84.9950	43.4347	24.1982
1							
	Linear	1480.4500	363.0796	83.7369	32.0067	13.9012	5.5209
	Nonlinear	2055.0010	503.9879	116.2345	44.4283	19.2961	7.6635
5							
	Linear	1242.0350	304.0540	69.5586	26.1335	10.9348	3.8999
	Nonlinear	1553.3360	380.2613	86.9926	32.6836	13.6754	4.8774

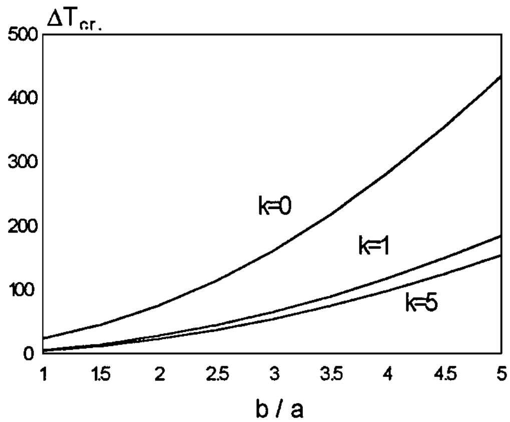

Fig. 3 Critical buckling temperature of the functionally graded plate under linear temperature change across the thickness vs b/a ($b/h = 100$).

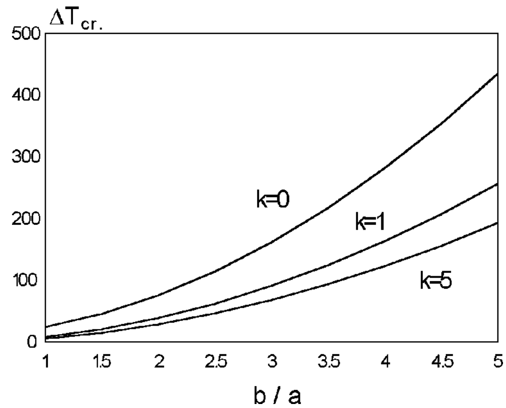

Fig. 5 Critical buckling temperature of the functionally graded plate under nonlinear temperature change across the thickness vs b/a ($b/h = 100$).

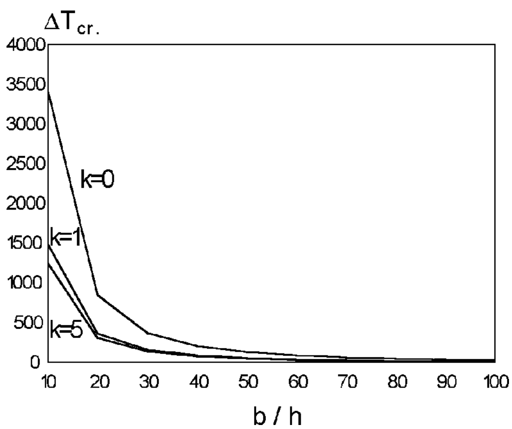

Fig. 4 Critical buckling temperature of the functionally graded plate under linear temperature change across the thickness vs b/h ($b/a = 1$).

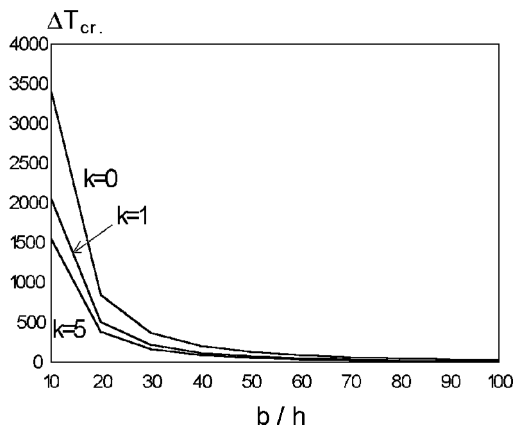

Fig. 6 Critical buckling temperature of the functionally graded plate under nonlinear temperature change across the thickness vs b/h ($b/a = 1$).

the power law index, $k = 0$, 1, 5, are considered. As explained earlier, variation of the composition of ceramics and metal is linear for $k = 1$. The value of $k = 0$ represents a homogeneous (fully ceramic) plate. Figures 1, 3, and 5 show that the buckling temperature increases by the increase of the dimension ratio b/a and decreases by the increase of the power law index k from 0 to 5. Figures 2, 4, and 6 show that the buckling temperature decreases by the increase of the dimension ratio b/h and the power law index k from 0 to 5.

Note that the buckling temperatures for homogeneous plates, $k = 0$, are considerably higher than those for the functionally graded plates, $k > 0$, especially for the comparatively longer and thicker plates. Comparing Figs. 3 and 4 with Figs. 5 and 6 shows, that the assumption of linear temperature distribution across the plate thickness estimates smaller values for ΔT_{cr} compared to the nonlinear temperature distribution. Tables 1 and 2 show a comparison of the results of buckling temperature between the linear and nonlinear temperature distributions across the plate thickness. Table 1 presents the results of buckling temperature for different values of k vs b/a. As k increases, the difference between the buckling temperature of linear and nonlinear temperature distributions increase. Similarly, the difference between two cases increases for longer b/a values (longer rectangular plates). Table 2 shows the buckling temperatures for different values of k vs b/h. As b/h is increased (thicker plates), the difference between the buckling temperature associated with the linear and nonlinear temperature distributions increases. Note that for the homogeneous plates, $k = 0$, the buckling temperatures obtained for linear and nonlinear temperature distributions are identical, as expected.

The buckling temperatures associated with the temperature difference across the thickness and along the length of the plate for homogeneous plates are identical when the same reference temperatures, $T_m = T_0$, are assumed. For FGM plates, the buckling temperature related to the linear temperature distribution along the length is higher than that of across the thickness.

Conclusions

In the present paper, equilibrium and stability equations for rectangular, simply supported, functionally graded plates are obtained. Derivation is based on the CPT, with the assumption of power law composition for the constituent materials. Then, the buckling analysis of functionally graded plates under four different types of thermal loadings are presented. Closed-form solutions for the critical buckling temperatures of plates are presented. The following conclusions are reached.

1) The equilibrium and stability equations are identical with the corresponding equations for homogeneous-isotropic plates.

2) The critical buckling temperature differences ΔT_{cr} for the functionally graded plates are generally lower than the corresponding values for homogeneous plates. Functionally graded plates have many advantages as a heat resistant material, but it is important to check their strength due to the thermal buckling.

3) The critical buckling temperature difference ΔT_{cr} for the functionally graded plates is reduced when the power law index k increases.

4) The critical buckling temperature difference ΔT_{cr} for the functionally graded plates is increased by increasing the dimension ratio b/a.

5) The critical buckling temperature difference for the functionally graded plates is decreased by increasing the dimension ratio b/h.

6) In functionally graded plates, the solution of the heat conduction equation results in a nonlinear temperature distribution across the thickness of the plate (the graded direction). The resulting buckling temperature is greater compared to the assumption of linear temperature distribution.

7) The critical buckling temperature difference ΔT_{cr} for the plates under temperature difference through the length is twice of the plates under uniform temperature rise $-2T_0$.

8) For the homogeneous plates, the critical buckling temperature difference ΔT_{cr} for the plate under temperature difference through the length is equal to the plate under temperature difference across the thickness when the same reference temperatures are considered, $T_m = T_0$.

9) For the functionally graded plates, the critical buckling temperature difference ΔT_{cr} for the plate under temperature difference along the length is greater than the plate under linear temperature difference across the thickness.

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